Computation of Output Feedback Gains for
Linear Stochastic Systems Using the Zangwill-Powell Method*

by

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Abstract

Because conventional optimal linear regulator theory results in a controller which requires the capability of measuring and/or estimating the entire state vector, it is of interest to consider procedures for computing controls which are restricted to be linear feedback functions of a lower dimensional output vector and which take into account the presence of measurement noise and process uncertainty. To this effect a stochastic linear model has been developed that accounts for process parameter and initial uncertainty, measurement noise, and a restricted number of measurable outputs. Optimization with respect to the corresponding output feedback gains was then performed for both finite and infinite time performance indices without gradient computation by using Zangwill's modification of a procedure originally proposed by Powell. Results using a seventh order process show the proposed procedures to be very effective.

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1. Introduction

Because conventional optimal linear regulator theory results in a controller which requires the capability of measuring and feeding back the entire state vector, it is of interest to consider procedures for computing controls which are restricted to be linear feedback functions of a lower dimensional output vector. Such a procedure, however, has its limitations in that the feedback gains will be functions of the initial state vector. In addition, the presence of measurement noise and process uncertainty can lead to additional problems relating to both modelling and computation.

To this effect a stochastic linear model has been developed that accounts for process parameter and initial uncertainty, measurement noise, and a restricted number of measurable outputs. Both finite and infinite time performance indices were considered. Optimization with respect to the output feedback gain was performed without gradient computation by using Zangwill's[1] modification of a procedure originally proposed by Powell[2]. This procedure is such that if the cost index were indeed quadratic in the gains, then the search would be along a set of conjugate directions.

The Zangwill-Powell method is especially useful for infinite time performance indices since many of the procedures proposed to date for finding output feedback gains for such indices cannot be guaranteed to converge to a solution.[3,4] Additional problems also arise if an intermediate gain perturbation result in an unstable system. This can be immediately corrected using the Zangwill-Powell procedure by setting the index itself to a very large number.

The effectiveness of the Zangwill-Powell algorithm was evaluated using sixth order linearized longitudinal equations of motion for an aircraft. Results showed the algorithms to be capable of converging to a set of gains useful for gust alleviation.

2. Problem Statement

The system being optimized is of the form:

Process: $\dot{x} = Ax + Bu + \Delta Ax + \Delta Bu + v(t)$ (1a)

Measurement: $y = Cx + n(t)$ (1b)

Control: $u = Ky$ (1c)

Initial State Covariance: $\mathcal{E}(x_0 \, x_0^T) = P_0$

where
x = $(n \times 1)$ state vector

u = $(\ell \times 1)$ control vector

y = $(m \times 1)$ output vector

K = gain matrix containing both fixed elements (most likely zero) and variable elements to be determined

v, n = white noise vectors with respective covariance matrices V, N.

and $\Delta A, \Delta B$ = uncertainty in A and B respectively.

Two procedures were considered in order to take into account the total process uncertainty $\Delta Ax + \Delta Bu$; namely:

1) Defining

$$w(t) = \Delta Ax(t) + \Delta Bu(t) \tag{2}$$

as an additional white noise vector with zero mean and assigned covariance matrix W suitably chosen to reflect the uncertainty. The procedures S developed by Joshi[5] are then applicable to solution.

2) Letting $\Delta Ax = \sum_i x_i \, \Delta a_i$ (3a)

and $\Delta Bu = \sum_i u_i \, \Delta b_i$ (3b) where Δa_i, Δb_i, the i^{th} columns of ΔA and ΔB respectively, are in turn set equal to

$$\Delta a_i = F_i \, w$$

$$\Delta b_i = G_i \, w$$

where w is a white noise vector with covariance matrix W, and F_i, G_i are constant matrices. The optimization procedure cited by McLane[6] are then applicable to the solution.

Given the above two formulations, the performance index to be minimized is:

$$J = \mathcal{E} \frac{1}{T} \int_0^T (y^T Q y + u^T ru) \, dt \tag{4}$$

where $\mathcal{E}$ denotes the statistical expectation operator.

Substituting eqs. 1b and 1c into J gives

$$J = \frac{1}{T} \int_0^T [x^T C^T Q C x + 2x^T C^T Q n$$

$$+ n^T Q n + x^T C^T K^T R K C x + 2 x^T C^T K^T R n + n^T K^T R K n] \, dt \tag{5}$$

This formulation of the index can be simplified by noting that

$$\mathcal{E}(x^T C^T Qn) = \mathcal{E}(x^T C^T K^T R n) = 0$$

Furthermore $(n^T Q n)$ is a constant term independent of the control and therefore has no effect on the index. Then minimization of (5) is equivalent to the minimization of

$$J = \mathcal{E} \frac{1}{T} \int_0^T x^T (C^T QC + C^T K^T R K C) \, x \, dt + \mathcal{E}(n^T K^T R K n) \tag{6}$$

Elimination of the expectation operator is now possible by recognizing that J can be rewritten as:

$$J = \text{Trace}\ (C^T Q C + C^T K^T R K C)\ S + \text{Trace}\ (K^T R K N) \tag{7}$$

where $$S = \frac{1}{T} \int_0^T \mathcal{E}\,(x\, x^T)\, dt \tag{8}$$

Thus upon computation of the integral of the state covariance matrix $\mathcal{E}(x\, x^T)$, the value of the index can be found from eq. 7.

In particular if T is finite, the covariance

$$P = \mathcal{E}(x\, x^T) \tag{9}$$

can be readily propogated, given a value for K, as follows:

- Formulation defined by eq. 2 [5].

$$\dot{P} = (A + B K C)\, P + P\, (A + B K C)^T + B K N K^T B^T + (V + W) \tag{10a}$$

$$P(0) = \mathcal{E}\,(x(0)\, x^T(0)) \tag{10b}$$

- Formulation defined by eq. 3 [6].

$$\dot{P} = (A + B K C)\, P + P(A + B K C)^T + M(P, K) + N(K)$$

$$+ B K N K^T B^T + V \tag{11a}$$

$$P(0) = \mathcal{E}\,(x(0)\, x^T(0)) \tag{11b}$$

where $$M(P,K) = \sum_{ij} P_{ij}\ \hat{F}_i\, W\, \hat{F}_j^{\,T} \tag{11c}$$

$$N(K) = \sum_{ij} G_i (K_i\, N\, K_j^T)\, W\, G_j^T \tag{11d}$$

$$\hat{F}_i = F_i + \sum_{\ell} G_\ell \, (K\, C)_{\ell i} \tag{11e}$$

$$K_i = i^{th} \text{ row of } K$$

and $$P_{ij} = i - j^{th} \text{ component of } P$$

For the case in which $T = \infty$, the integral S of eq. 8 will not in general converge if there is measurement noise (n) and/or process noise (v). Thus as in refs. 3, 4, 5, in the limit S will be replaced by

$$S = \text{Lim} \; \frac{1}{T} \int_0^T P(t)\, dt = P_{ss} \tag{12}$$

where P_{ss} is the steady state solution to either eq. 10 or eq. 11.

3. Computational Procedures

Since the performance index (eq. 7) is easily evaluated given a value for the gain matrix K, the Zangwill-Powell[1,2] method which does not require gradient computation is very attractive for optimization. In particular, the IMSL subroutine ZXPOWL was used for implementation.[7] Starting with initial values for elements of the gain matrix K, successive perturbations are made in each of the variable elements and the corresponding value of J computed. Using the computed indices, perturbation directions are chosen such that convergence to the minimum of a quadratic function requires a finite number of iterations. One particular attractive feature of the algorithms is the ability to correct, when $T = \infty$, for a set of unstable gains which do not permit the determination of a steady state covariance matrix, P. This was done by computing the eigenvalues of $(A + B\, K\, C)$ for each perturbed value of K and setting J equal to a very large number (i.e., 10^{50}) whenever instability is noted.

4. Experimental Results

4.1 System definition

A modified 6-dimensional version of the TIFS[8] aircraft with a gust input (σ = 15 fps) was used for evaluation in the presence of a zero reference command. The corresponding variable definitions were as follows:

Plant state:

$$\underline{x} = \begin{pmatrix} q \\ \Delta\theta \\ \Delta V \\ \Delta\alpha \\ \delta e \\ \delta z \\ \alpha_g \end{pmatrix} = \begin{pmatrix} \text{pitch rate} \\ \text{pitch angle} \\ \text{velocity} \\ \text{angle of attack} \\ \text{elevator deflection} \\ \text{direct lift flap deflection} \\ \text{gust induced attack angle} \end{pmatrix}$$

Plant control:

$$\underline{u} = \begin{pmatrix} \delta_{ec} \\ \delta_{zc} \end{pmatrix} \cdot \begin{pmatrix} \text{elevator command} \\ \text{lift flap command} \end{pmatrix}$$

Observations:

$$\underline{y} = \begin{pmatrix} q \\ \Delta\theta \\ \alpha \\ nz_1 \\ nz_2 \\ \dot{\gamma} \end{pmatrix} = \begin{pmatrix} \text{pitch rate} \\ \text{pitch angle} \\ \text{angle of attack} \\ \text{Point 1 vertical acceleration} \\ \text{Point 2 vertical acceleration} \\ \text{flight path angular rate} \end{pmatrix}$$

The structural matrices corresponding to climb condition, i.e., h = 1524 m, V = 106 m/s, which were used for evaluation purposes are (see eq. 1):

$$A = \begin{pmatrix} -.1686 & .000035 & .000231 & -.486 & -4.3778 & -.19948 & -.486 \\ 1. & 0. & 0. & 0. & 0. & 0. & 0. \\ 0. & -32.17 & -.0143 & 18.027 & 0. & -3.0933 & .0518 \\ 1. & 0.000013 & -.000531 & -1.223 & -.1273 & -.2667 & -1.223 \\ 0. & 0. & 0. & 0. & -20. & 0. & 0. \\ 0. & 0. & 0. & 0. & 0. & -40. & 0. \\ 0. & 0. & 0. & 0. & 0. & 0. & -.2784 \end{pmatrix}$$

$$B = \begin{pmatrix} 0. & 0. \\ 0. & 0. \\ 0. & 0. \\ 0. & 0. \\ 20. & 0. \\ 0. & 40. \\ 0. & 0. \end{pmatrix}$$

$$C = \begin{pmatrix} 1. & 0. & 0. & 0. & 0. & 0. & 0. \\ 0. & 1. & 0. & 0. & 0. & 0. & 0. \\ 0. & 0. & 0. & 1. & 0. & 0. & 0. \\ 64.63 & .00318 & .176 & 444.2 & 212.1 & 100.4 & 444.2 \\ -61.82 & .00580 & .193 & 407.8 & -116.2 & 85.5 & 407.8 \\ 0. & .000013 & .000531 & 1.223 & .1273 & .2667 & 1.223 \end{pmatrix}$$

Corresponding sensor noise deviations were:

$$\sigma_q = .5 \text{ deg/sec}$$

$$\sigma_\theta = 0.2 \text{ deg}$$

$$\sigma_{n_z} = 0.05 \text{ g}$$

Since the only plant disturbance was the excitation for the gust $\mathcal{E}(v_i^2) = 0 \quad i = 1\text{-}6$; to account for a 15 fps standard deviation $\mathcal{E}(v_7^2)$ was set equal to .0003.

Values for q_{ii} and r_{ii} in eq. 4 were chosen to be representative of the inverse maximum squared value of the weighted variables. In particular the following values were used:

$$q_{11} = 2500.$$
$$q_{22} = 50.$$
$$q_{33} = 50.$$
$$q_{44} = 4.$$
$$q_{55} = 4.$$
$$q_{66} = 2500.$$
$$r_{11} = 6.$$
$$r_{22} = 3.$$

Also considered was the situation in which no penalty was placed on n_{z1}, n_{z2}, and $\dot{\gamma}$ (i.e., $q_{44} = q_{55} = q_{66} = 0$).

For the stochastic problem defined by eq. 2, the covariance W of the plant disturbance was chosen to be

$$\text{Diag } (.2, 0., .0007, .0005, 0., 0.)$$

These elements were chosen by computing for each component of the state equation

$$W(i,i) \simeq \sum_j (\Delta a_{ij}^2) \times (\text{MAX } (x_j^2))$$

where Δa_{ij} was approximated to reflect the data in reference 8.

For the problem defined by eq. 3, the G_i's were set equal to zero, and the F_i's were selected such that the standard deviations in the corresponding components of A were (in matrix form):

$$\begin{pmatrix} .4 & 0 & .002 & .2 & 2.2 & .11 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & .001 & .17 & 0 & .025 & 0 \\ 0 & 0 & 0 & .24 & .024 & .16 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

4.2 Results

Using the preceding data, the six gains relating $q, \Delta\theta, \Delta\alpha$ to δe_c and δz_c were determined.

Of interest were the following observations:

- Convergence time for the infinite time problem was an order of magnitude less than the time required for optimization over a two second interval. This resulted from having to compute only the solution to a set of algebraic equations rather than a set of differential equations.
- Assuming that every 6 gain perturbations called for by the Zangwill-Powell method corresponds to a gradient evaluation, it was noted that the number of gradient evaluations performed by a steepest descent algorithm was comparable to the number performed by the Zangwill-Powell method.
- For a finite time index it is indeed possible to obtain gains that yield an unstable set of eigenvalues. This resulted when the finite time version of the formulation defined by eq. 2 was used with no weighting on n_{z1}, n_{z2}, $\dot{Y}$.

In order to test performance with respect to the behavior of the states and the control signals, the gains were used to regulate the longitudinal motion subject to the initial condition:

$$p = .02r/s, \Delta\theta = .15r, \Delta\alpha = .15r$$

Figure 1 depicts responses for q, α, and n_{z1} which resulted from applying those gains which resulted from the finite time version of the formulation defined by eq. 2 when $W = 0$ and with no weighting applied to n_{z1}, n_{z2}, $\dot{\gamma}$.

5. Conclusions

Optimal output feedback control gains were determined using the Zangwill-Powell procedure which does not require gradient computation. Two stochastic formulations of the process equation were considered in order to take into account process uncertainty, process disturbances and sensor noise. Results using a seventh order system show that the Zangwill-Powell method is very effective for control gain computation.

Continuing efforts are considering such topics as:

- Constraints on the gain magnitudes.
- Effects of reference commands.
- Evaluation of the two procedures for modelling process uncertainties.

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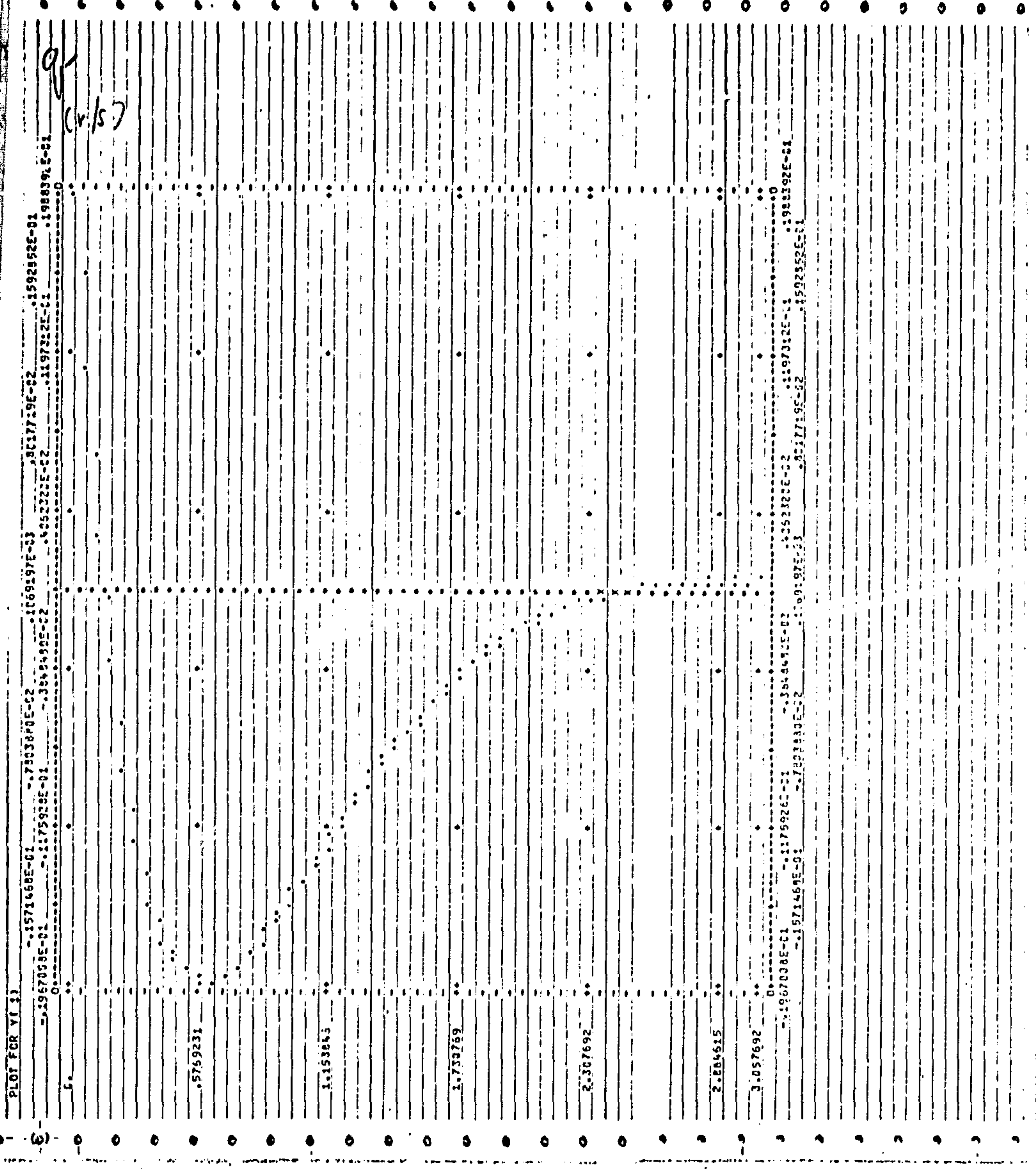

Fig. 1a q response

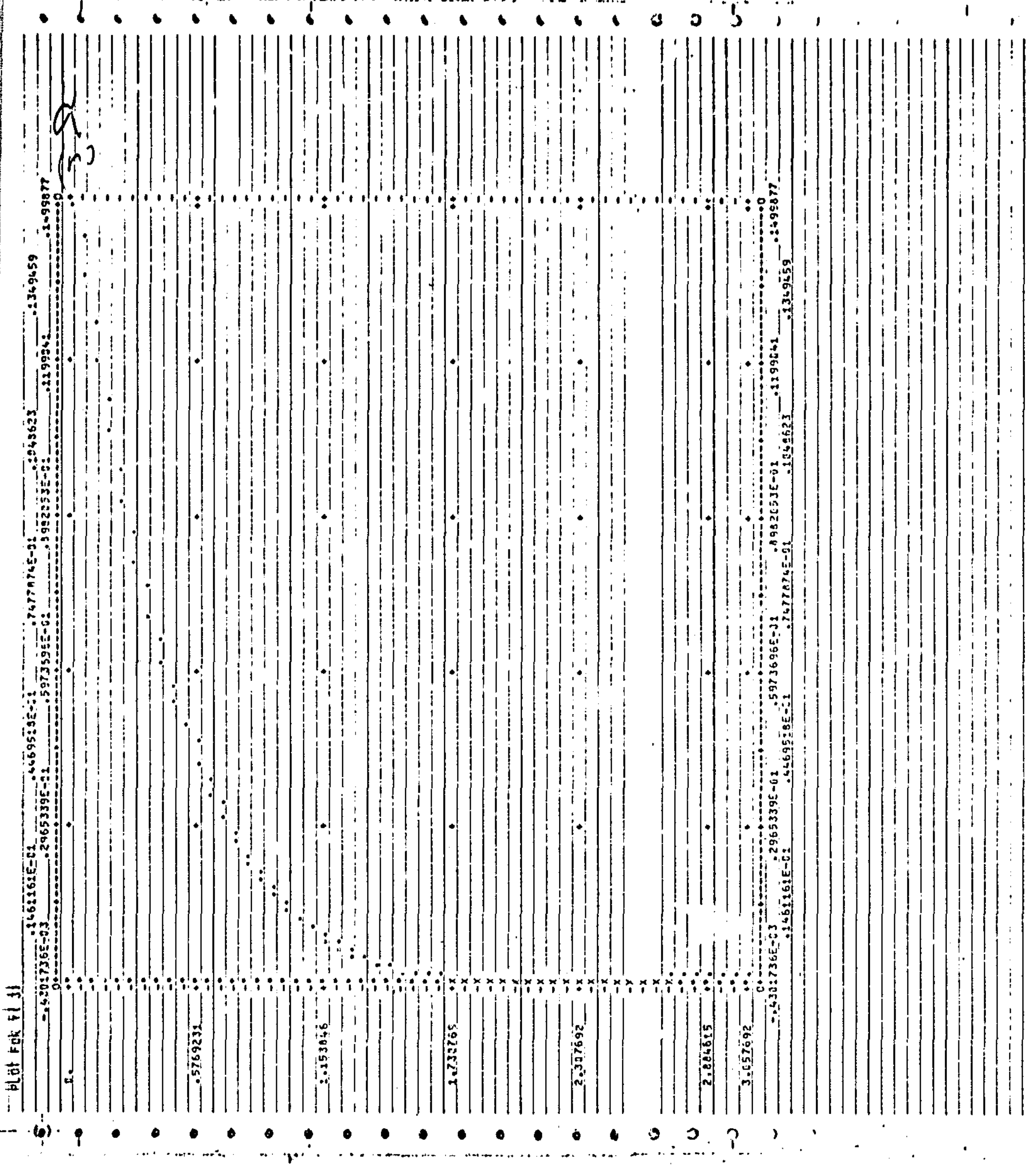

Fig. 1b α response

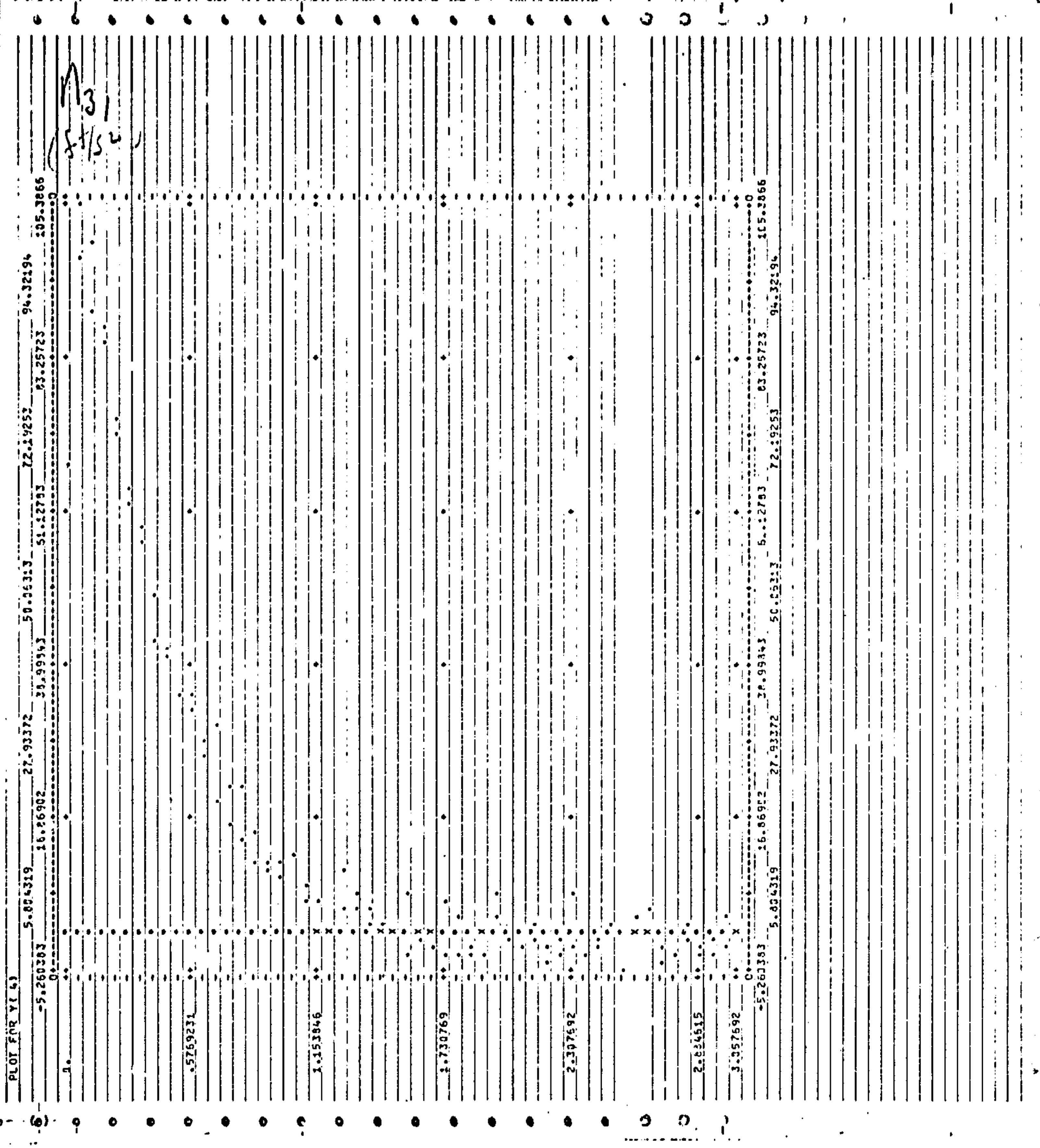

Fig. 1c n_z response